Calculus of Noise

23.05.02

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Calculus of Noise

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Introduction

Introduction

Motivation for Stochastic Caculus; Population Growth Model

Simple ODE problem depicting the growth of population N over time t

$$\frac{dN}{dt}(t) = a(t)N(t), \;\; N(0) = N_0$$

where $a(t)$ is called the *relative growth rate*

Introduction

(Solution)

Suppose $a(t)$ is *deterministic* (i.e. $a(t)=a$ is a constant).

$$\int_0^t a\,ds = \int_0^t \frac{\frac{dN}{dt}(s)}{N(s)}ds = \ln\left(\frac{N(t)}{N_0}\right)$$

$$\int_0^t a\,ds = at$$

$$\ln\left(\frac{N(t)}{N_0}\right) = at,\ \ N(t) = N_0 e^{at}$$

■

Introduction

Motivation for Stochastic Caculus; How to integral a function with a noise?

In real world, *the relative growth rate* $a(t)$ can be "stochastic".

Thus, we can use the expression

$$a(t) = r(t) + j(t) \cdot \text{(random error)}$$

Introduction

The problem can be rewritten as SDE (stochastic differential equation)

$$\frac{dN}{dt}(t) = (r(t) + j(t) \cdot \text{noise})N(t)$$

$$\frac{dN}{dt}(t) = r(t) \cdot N(t) + j(t) \cdot \text{noise} \cdot N(t)$$

Using integration expression,

$$N(t) = N_0 + \int_0^t r_s N_s \, ds + \int_0^t j_s N_s \cdot noise \cdot ds$$

but, how do we integrate the term?

$$\int_0^t j_s N_s \cdot noise \cdot ds$$

Introduction

Solution; The Ito's Integral

We will define the new integration

$$\int_0^t j_s N_s \bullet noise \bullet ds = \int_0^t j_s N_s \bullet dB_s$$

to make ous feel that dB_s has a similar meaning to $noise \bullet ds$ and $B_{s+\epsilon} - B_s$

where B_s is a *Brownian motion* which is a "*stochastic process*"

which is strongly related with normal distribution

Background

Background

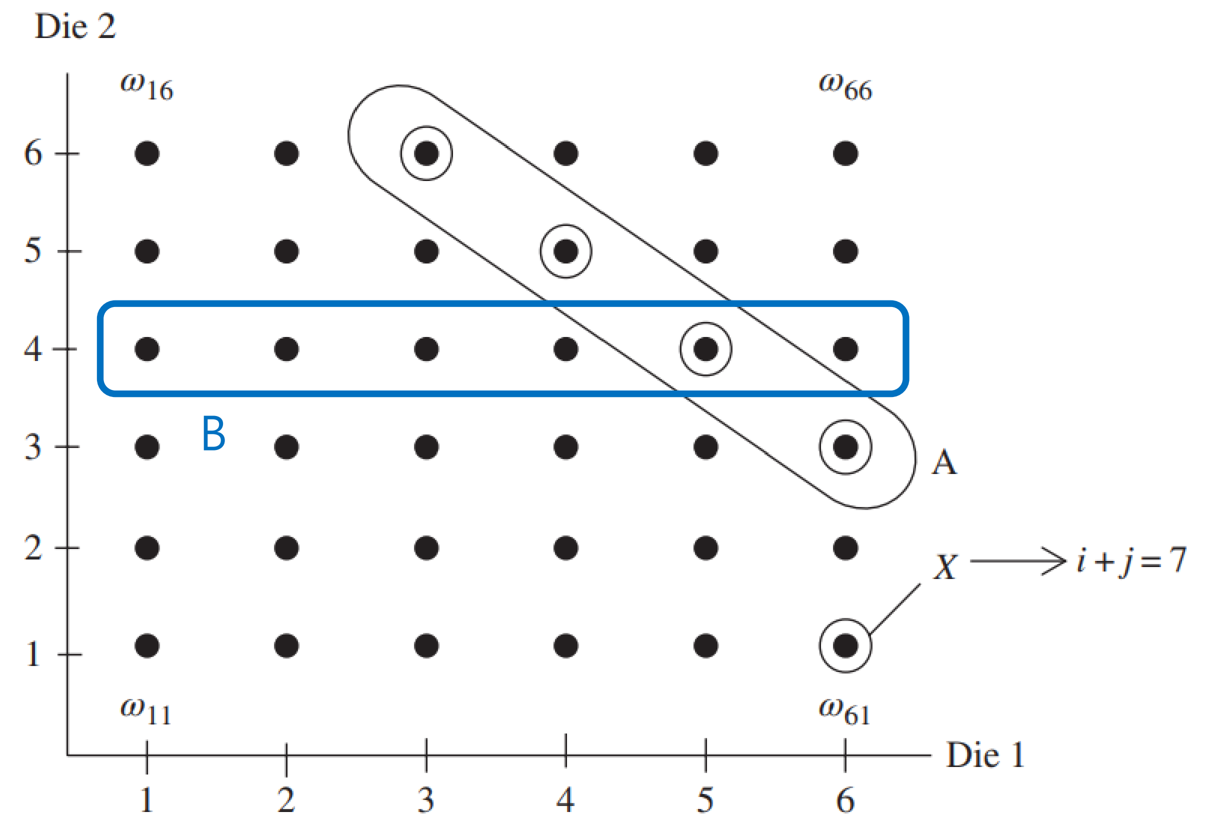

Figure 2.1 Event $A = \{\omega_{36}, \omega_{45}, \omega_{54}, \omega_{63}\}$.

Ω "All outcomes"

$A, B \in \mathcal{A}$ "Events"

$\boldsymbol{P}(A)$ "The probability" of A

A $\boldsymbol{\sigma}$-algebra

Background

Definition (σ-algebra)

(Setting)

$$\Omega \text{ : some set}$$

$$P(\Omega) \text{ : the power set of } \Omega$$

A ***σ**-algebra* (or ***σ**-field*) is a subset $\mathcal{A}$ of $P(\Omega)$ that satisfies

(1) $\Omega \in \mathcal{A}$

(2) If $A \in \mathcal{A}$, then the complement $A^C \in \mathcal{A}$

(3) $\mathcal{A}$ is closed under countable union. i.e.

$$A_1, \dots, A_k, \dots \in \mathcal{A} \Rightarrow \sum_{k=1}^{\infty} A_k \in \mathcal{A}$$

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Background

Definition (Measurable function)

(Setting)

$$(X,\mathcal{A}) \ \& \ (Y,\boldsymbol{B}) : \text{measurable spaces}$$

A ($\mathcal{A}$-$\mathcal{G}$) *measurable* function is a mapping $f: X \to Y$ s.t.

$$f^{-1}(B) \in \mathcal{A} \text{ for every set } B \in \boldsymbol{B}.$$

If $f: X \to Y$ is a measurable function, then one can write

$$f : (X,\mathcal{A}) \longrightarrow (Y,\boldsymbol{B})$$

to emphasize the dependency on two $\boldsymbol{\sigma}$-algebras $\mathcal{A}$ & $\boldsymbol{B}$

In special case, when $(Y,\mathscr{B})$ is a topological space with the Borel $\boldsymbol{\sigma}$-algebra $\mathscr{B}$, we say

$$f \text{ is } \mathcal{A}\text{-}\textit{measurable} \text{ if } f^{-1}(B) \in \mathscr{B} \text{ for every set } B \in \mathscr{B}$$

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Background

Notation

Note that

$$A \in \boldsymbol{\mathcal{A}} \Leftrightarrow \text{The indicator function } 1_A \text{ is measurable w.r.t. a } \boldsymbol{\sigma}\text{-field } \boldsymbol{\mathcal{A}}$$

We use notation

$$X \in \boldsymbol{\mathcal{A}} :\Leftrightarrow \text{The function } X \text{ is measurable w.r.t. a } \boldsymbol{\sigma}\text{-field } \boldsymbol{\mathcal{A}}$$

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Background

Definition (The $\boldsymbol{\sigma}$-algebra generated by functions)

(Setting)

$f : X \to Y$ be a function

$F = \{f_i\}_{i \in I}$ be a collection of functions $f_i : X \to Y$

B be a $\boldsymbol{\sigma}$-algebra of subsets of Y

Define *the $\boldsymbol{\sigma}$-algebra generated by f* as

the smallest $\boldsymbol{\sigma}$-algebra on X containing $\{f^{-1}(S) : S \in B\}$

denoted by $\boldsymbol{\sigma}(f)$

Define *the $\boldsymbol{\sigma}$-algebra generated by $F = \{f_i\}_{i \in I}$* as

the smallest $\boldsymbol{\sigma}$-algebra on X containing $\{f_i^{-1}(S) : S \in B\}$ for all $i \in I$

denoted by $\boldsymbol{\sigma}(F)$

Moreover,

a function $f : X \to Y$ is measurable w.r.t the $\boldsymbol{\sigma}$-algebra $\mathcal{C}$ of $X \Leftrightarrow \boldsymbol{\sigma}(f) \subseteq \mathcal{C}$

Background

Definition (Measure)

(Setting)

$$\Omega \text{ : some set}$$

$$\boldsymbol{\mathcal{A}} \text{ : a } \boldsymbol{\sigma}\text{-}\textit{algebra} \text{ (or } \boldsymbol{\sigma}\text{-}\textit{field} \text{) of } \Omega$$

A function $P:\boldsymbol{\mathcal{A}}\to\mathbb{R}_{\infty}$ is called a *measure* if it satisfies the following properties

(1) Non-negativity

$$P(A)\geq 0 \text{ for all } A\in\boldsymbol{\mathcal{A}}$$

(2) Null-empty set

$$P(\phi)=0$$

(3) $\boldsymbol{\sigma}$-additivity

For pairwise disjoint sets $A_1, \ldots, A_k, \ldots \in\boldsymbol{\mathcal{A}}$

$$P\left(\sum_{k=1}^{\infty} A_k\right)=\sum_{k=1}^{\infty} P(A_k)$$

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Background

Definition (Probability Space)

(Setting)

$$\Omega \text{ : the sample space of the random experiment}$$

$$\boldsymbol{\mathcal{A}} \text{ : a } \boldsymbol{\sigma}\text{-algebra of } \Omega$$

$$\text{i.e. } (\Omega, \boldsymbol{\mathcal{A}}) \text{ is a measurable space}$$

A measure $\boldsymbol{P}: \boldsymbol{\mathcal{A}} \to \mathbb{R}_{\infty}$ is called a *probability measure* on a *probability space* $(\Omega, \boldsymbol{\mathcal{A}})$

$$\text{if } P(\Omega) = 1$$

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Background

Definition (a Complete Probability Space)

A probability space $(\Omega, \boldsymbol{\mathcal{A}}, \mathcal{P})$ is *complete* when

$$\{E \subseteq \Omega : P^*(E) = 0\} \subseteq \boldsymbol{\mathcal{A}}$$

where $P^*(E) := \inf\{P(F) : E \subseteq F \ \& \ F \in \boldsymbol{\mathcal{A}}\}$

Background

Definition (Random Variable)

A (real) *random variable* X on (Ω, $\boldsymbol{\mathcal{A}}$) is a measurable function from Ω to $\mathbb{R}$

i.e. $\{\ X^{-1}(B) : B \in \mathscr{B}\ \} \subseteq \boldsymbol{\mathcal{A}}$

where $\mathscr{B}$ is the Borel $\boldsymbol{\sigma}$-algebra on $\mathbb{R}$

Denote the probability that X takes on a value $X(w)$ in a mesurable set $B \in \mathscr{B}$ as

$$P(X \in B) := P(\{w \in \Omega : X(w) \in B\})$$

Background

Definition (Expectation)

(Setting)

$$(\Omega,\ \mathcal{A},\ \mathcal{P}) : \text{a probability space}$$

$$X : \text{a random variable on } (\Omega,\ \mathcal{A})$$

If X is integrable, then the expectation of X is defined by

$$E(X) := \int_{\Omega} X \, dP$$

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Background

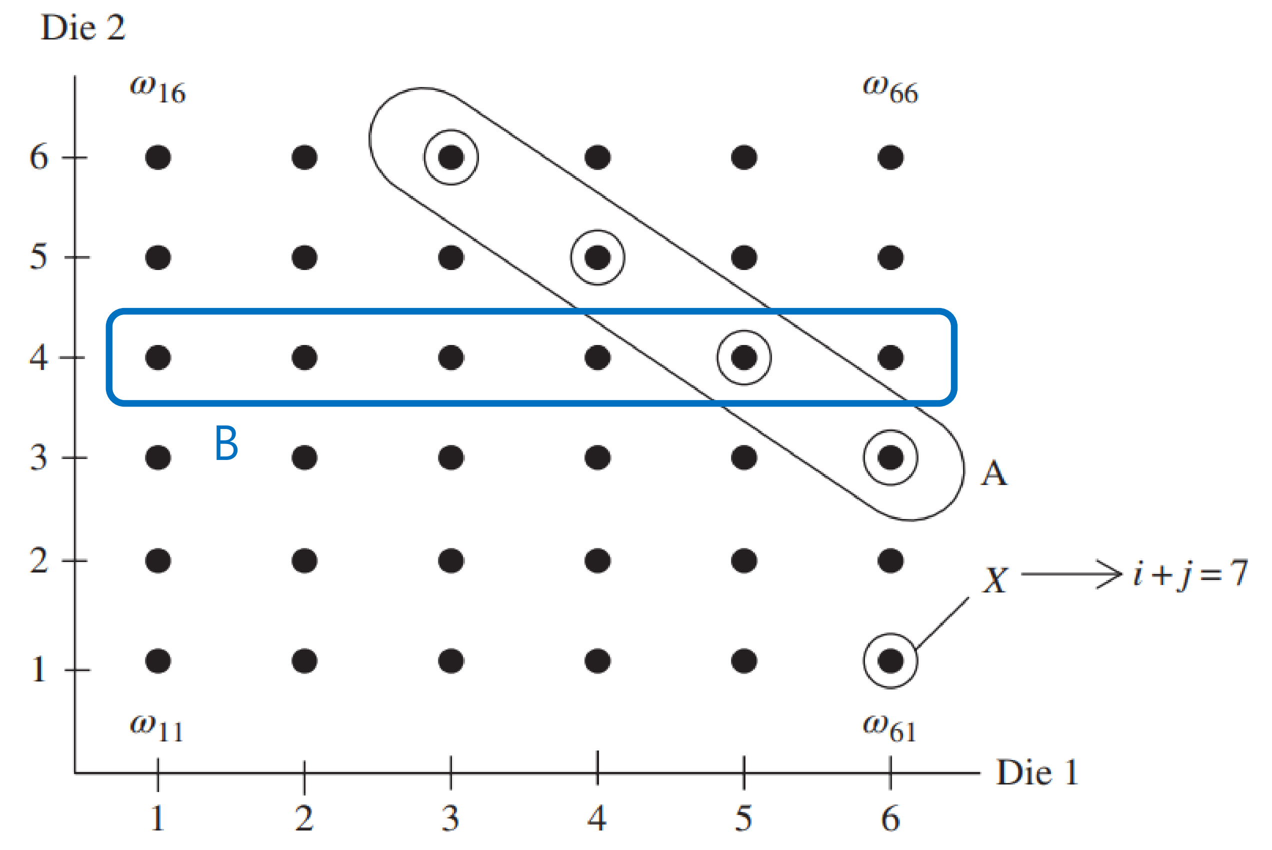

Figure 2.1 Event $A = \{\omega_{36}, \omega_{45}, \omega_{54}, \omega_{63}\}$.

Brownian Motion

Brownian Motion

Definition (Stochastic Process)

A *stochastic process* $\{X_t\}_{t \in T}$ is a parametrized collection of random variables which are defined on a given probability space (Ω, $\mathcal{A}$, $\boldsymbol{P}$).

In discrete time, take $T = Z_{\geq 0}$, or if we work in contunuous time, set $T = [0, \infty)$

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Brownian Motion

Definition (Sample Path)

For each fixed $t \in T$, we have a random variable $X_t : w \in \Omega \mapsto X(w) = X(t, \bullet)$

For each fixed $w \in \Omega$, we just have a real valued fuction $t \in T \mapsto X_t(w)$

$X_t(w)$ is called *the sample path* of the process for fixed $w \in \Omega$

Brownian Motion

Definition (Filtration)

A *filtration* on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ is a collection $F = \{F_t\}_{t \in T}$ of sub-$\boldsymbol{\sigma}$-fields of a $\boldsymbol{\sigma}$-field $\mathcal{A}$ having the property that

$$s \le t \text{ implies } F_s \subseteq F_t$$

A *filtered probability space* $(\Omega, \mathcal{A}, F, \mathcal{P})$ is a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ endowed with a filtration $F = \{F_t\}_{t \in T}$

We say that a stochastic process $\{X_t\}_{t \in T}$ is *adapted* to the filtration $F = \{F_t\}_{t \in T}$

$$\text{when } X_t \text{ is } F_t\text{-measurable for each } 0 \le t < \infty$$

A stochastic process $\{X_t\}_{t \in T}$ is always adapted to its *natural filtration* $F_t^X = \sigma(X_s : s \le t)$ that is the smallest filtration to which $\{X_t\}_{t \in T}$ is adapted.

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Brownian Motion

Definition (Standard) Brownian Motion

The *standard Brownian motion* is a continuous-time stochastic process $\{B_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ with the following properties:

① $P(B_0 = 0) = 1$

② For $0 = t_0 \leq t_1 \leq \cdots \leq t_k$,

the increments (displacements) $B(t_1),\ B(t_2) - B(t_1),\ \cdots\ ,\ B(t_k) - B(t_{k-1})$

are independent random variables.

③ For $0 \leq s \leq t$, the increments $B_t - B_s \sim N(0, t-s)$

Brownian Motion

Definition (the Standard Brownian Filtration)

(Setting)

$\{B_t\}_{t\in[0,T]}$ be a Brownian motion defined on a probability space (Ω, $\mathcal{A}$, $\mathcal{P}$).

$\boldsymbol{\sigma}\{B_s\}_{s\le T}$: the $\boldsymbol{\sigma}$-field generated by $\{B_t\}_{t\in[0,T]}$

$$N_T := \{B\in\sigma\{B_s\}_{s\le T} : P(B)=0\}$$

$N := \{A\subset\Omega : A\subseteq B \text{ for some } B\in N_T\}$, called "*the set of null sets*"

Assume that the probability measure P is extended so that $P(A)=0$ for each $A\in N$

Define *the standard Brownian filtration* as

the filtration $\{F_t\}_{t\in[0,T]}$

where F_t is the smallest $\boldsymbol{\sigma}$–field containing both $\boldsymbol{\sigma}\{B_s\}_{s\le t}$ and N

■

Brownian Motion

Definition (the usual condition)

The standard Brownian filtration $\{F_t\}_{t\in[0,T]}$ has two basic properties

(1) $\{F_t\}_{t\in[0,T]}$ is a *Right-continuous* filtration.

$$F_t = F_{t+} := \bigcap_{t<s\le T} F_s \text{ , for each } t\in[0,T]$$

(2) $\{F_t\}_{t\in[0,T]}$ is a *complete* filtration.

$$N\in F_t \text{ , for each } t\in[0,T]$$

We say "the standard Brownian filtration satisfies the *usual conditions*."

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Stochastic Integral

Stochastic Integral

Definition (the natural domation of Ito Integral)

(Setting)

$(\Omega, \boldsymbol{\mathcal{A}}, \boldsymbol{P})$ is a complete probability space with a the standard Brownian filtration $\{F_t\}$

B :the set of Borel sets of $[0, T]$

Let $H^2([0, T])$ be the space of stochastic process $f = \{f_t(w)\}_{t \in [0, T]}$ s.t.

(1) (Measurable)

$f_t : \Omega \times [0, T] \rightarrow (\mathbb{R}, B)$ is measurable w.r.t the $\boldsymbol{\sigma}$-algebra $F_T \times B$ on $\Omega \times [0, T]$.

$\Leftrightarrow f(\bullet, \bullet) \in F_T \times B$ for all $t \in [0, T]$

(2) (Adapted)

The random variable $f(\bullet, t)$ is adapted to $\{F_t\}$

$\Leftrightarrow f(\bullet, t)$ is $\boldsymbol{\mathcal{A}}_t$-measurable for each $t \in [0, T]$

$\Leftrightarrow f(\bullet, t) \in F_t$ for all $t \in [0, T]$

(3) (Integrability Constraint)

$$\| f \|_2([a,b]) := E\left(\int_a^b |f(w,t)|^2 dt \right)^{\frac{1}{2}} < \infty$$

$H^2([0, T])$ is called *the natural domain of Ito integral*

■

Stochastic Integral

Notation

Let $H_0^2([0,T])$ be the subspace of $H^2([0,T])$ which consists $f=\{f_t(w)\}_{t\in[0,T]}$ s.t.

$$f(w,t)=\sum_{i=0}^{n-1}a_i(w)1_{(t_i<t<t_{i+1})}(t)$$

where $0=t_0<t_1<\cdots<t_{n-1}<t_n=T$, $E(a_i^2)<\infty$ & $a_i\in F_{t_i}$

(i.e. a random variable a_i is F_{t_i}-measurable)

Stochastic Integral

Definition

For $f \in H_0^2([0,T])$, define Integral of f as

$$(I(f))(w) := \sum_{i=0}^{n-1} a_i(w)\{B_{t_{i+1}} - B_{t_i}\}$$

Note that $(I(f))(w)$ is just a *random variable* even though f is a *process*.

Stochastic Integral

Lemma 6.1. (Ito's Isometry on H_0^2)

For $f \in H_0^2([0,T])$,

$$\| I(f) \|_{L^2(dP)} = \| f \|_{L^2(dP \times dt)}$$

Stochastic Integral

(Proof)

We prove $\| I(f) \|^2_{L^2(dP)} = \| f \|^2_{L^2(dP\times dt)}$.

Since $f(w,t)=\sum_{i=0}^{n-1} a_i(w) 1_{(t_i < t < t_{i+1})}(t) \Rightarrow f^2(w,t)=\sum_{i=0}^{n-1} a_i^2(w) 1_{(t_i < t < t_{i+1})}(t)$,

$$\text{R.H.S} = E\left[\int_0^T f^2(w,t)dt\right] = E\left[\int_0^T \left(\sum_{i=0}^{n-1} a_i^2(w) 1_{(t_i < t < t_{i+1})}\right)dt\right] \quad (\because \text{Definition})$$

$$= E\left[\sum_{i=0}^{n-1} a_i^2(w) \int_0^T 1_{(t_i < t < t_{i+1})}(t)\right] = \sum_{i=0}^{n-1} E\left[a_i^2(w)\right](t_{i+1} - t_i)$$

Stochastic Integral

$$\text{L.H.S} = E\left[I(f)^2\right] = E\left[\sum_{i=0}^{n-1} a_i^2(w)\left\{B_{t_{i+1}} - B_{t_i}\right\}^2\right] \quad (\because \text{Definition})$$

$$= \sum_{i=0}^{n-1} E\left[a_i^2(w)\right] E\left[\left(B_{t_{i+1}} - B_{t_i}\right)^2\right] \quad (\because a_i \text{ and } B_{t_{i+1}} - B_{t_i} \text{ are independent})$$

$$= \sum_{i=0}^{n-1} E\left[a_i^2(w)\right]\left(t_{i+1} - t_i\right)$$

$$(\because B_{t_{i+1}} - B_{t_i} \sim N(0, t_{i+1} - t_i) \ \& \ V(X) = E(X^2) - (E(X))^2)$$

$\blacksquare$

Stochastic Integral

Lemma (H_0^2 is dence in H^2)

For any $f \in H^2([0,T])$, there exists a seq. $f_n \in H_0^2([0,T])$ s.t.

$$\| f - f_n \|_{L^2(dP \times dt)} \to 0 \text{ as } n \to \infty$$

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Stochastic Integral

By the last Lemma, for any $f \in H^2([0,T])$, define

$$I(f) = \lim_{n\to\infty} I(f_n)$$

(Existence)

Suppose there exists a seq. $f_n \in H_0^2([0,T])$ s.t. $\| f - f_n \|_{L^2(dP \times dt)} \to 0$ as $n \to \infty$.

$\Rightarrow$ a sequence f_n is a Cauchy sequence.

$\Rightarrow$ By Ito isometry, a sequence $I(f_n)$ is a Cauchy sequence in $L^2(dP)$

$\Rightarrow$ $L^2(dP)$ is a complete metric space i.e. every Cauchy sequences converge.

$\therefore$ $I(f_n)$ converges to some element of $L^2(dP)$

Stochastic Integral

(Well-defined)

Suppose f'_n is anoter sequence s.t. $\| f - f'_n \|_{L^2(dP\times dt)} \to 0$ as $n \to \infty$.

$\Longrightarrow$ By triangle inequality, $\| f_n - f'_n \|_{L^2(dP\times dt)} \to 0$

$\Longrightarrow$ By Ito isometry, $\| I(f_n) - I(f'_n) \|_{L^2(dP)} \to 0$ as $n \to \infty$

$$\therefore\ I(f_n) = I(f'_n)$$

$\blacksquare$

Stochastic Integral

Theorem (Ito's Isometry on H^2)

For $f \in H^2([0,T])$,

$$\| I(f) \|_{L^2(dP)} = \| f \|_{L^2(dP \times dt)}$$

Stochastic Integral

(Proof)

Since H_0^2 is dence in H^2, choose a seq. $f_n \in H_0^2([0,T])$ s.t.

$$\| f - f_n \|_{L^2(dP\times dt)} \to 0 \text{ as } n \to \infty$$

By the triangle inequality,

$$\| f \|_{L^2(dP\times dt)} \le \| f - f_n \|_{L^2(dP\times dt)} + \| f_n \|_{L^2(dP\times dt)}$$

By taking the limit,

$$\| f \|_{L^2(dP\times dt)} \le \| f_n \|_{L^2(dP\times dt)} \text{ as as } n \to \infty$$

Similary,

$$\| f_n \|_{L^2(dP\times dt)} \le \| f_n - f \|_{L^2(dP\times dt)} + \| f \|_{L^2(dP\times dt)}$$

$$\| f_n \|_{L^2(dP\times dt)} \le \| f \|_{L^2(dP\times dt)} \text{ as as } n \to \infty$$

$$\therefore \ \| f_n \|_{L^2(dP\times dt)} \to \| f \|_{L^2(dP\times dt)} \text{ as as } n \to \infty$$

Stochastic Integral

By the same way, since $I(f) = \lim_{n \to \infty} I(f_n)$ in $L^2(dP)$,

$$\therefore \; \| I(f_n) \|_{L^2(dP)} \to \| I(f) \|_{L^2(dP)} \text{ as as } n \to \infty$$

By Ito's Isometry on H_0^2,

$$\| I(f_n) \|_{L^2(dP)} = \| f_n \|_{L^2(dP \times dt)}$$

Therefore, $\| I(f) \|_{L^2(dP)} = \| f \|_{L^2(dP \times dt)}$

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Stochastic Integral

Ito's Integral; From a random variable to a stochastic process

We construct the map $I: H^2 \rightarrow L^2(dP)$, but to represent stochastic process, we need a mapping that takes *a process* to *a process* – not to a random variable.

For this purpose, we define a trancation function $m_t(w,s) \in H^2([0,T])$ defined by

$$m_t(w,s) = \begin{cases} 1 & \text{if } s \in [0,t] \\ 0 & \text{otherwise} \end{cases} \text{ for } t \in [0,T]$$

For $f \in H^2([0,T])$, the product $m_t f \in H^2([0,T])$ for all $t \in [0,T]$, so $I(m_t f)$ is a well-defined element of $L^2(dP)$

Stochastic Integral

Definition (Martingale)

(Setting)

$\{F_t\}_{0 \le t}$: a filtration on the probability space (Ω, $\mathcal{A}$, $\mathcal{P}$)

Suppose a stochastic porcess $\{X_t\}_{0 \le t}$ is adapted to the filtration $\{F_t\}_{0 \le t}$

(or $X_t \in F_t$ i.e. X_t is F_t-measurable for each $0 \le t$)

We say $\{X_t\}_{0 \le t}$ is a *continuous-time martingale* if

$$E(X_t \mid A_s) = X_s \text{ for any } 0 \le s \le t$$

$\{X_t\}_{0 \le t}$ is called a *(sub/super) martingale* when "=" is replaced by "$\le$" or "$\ge$"

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Stochastic Integral

Definition (Maximal Sequence of Martingale)

(Setting)

$\{M_n\}$: any sequence of random variables

The sequence defined by

$$M_n^* := \sup\{M_m : 0 \le m \le n\}$$

is called *the maximal sequence associated with* $\{M_n\}$

■

Stochastic Integral

Theorem (Doob's Maximal Inequality)

If $\{M_n\}$: a non-negative submartingale & $\lambda > 0$, then

$$\lambda P(M_n^* \geq \lambda) \leq E[M_n]$$

■

Stochastic Integral

Theorem (Ito Integrals as Martingales)

For any $f \in H^2[0, T]$, there is a process $\{X_t\}_{t \in [0, T]}$

(1) $\{X_t\}_{t \in [0, T]}$ is a continuous-time martingale w.r.t. the standard Brownian filtration F_t

(2) For each $t \in [0, T]$, $P(\{w : X_t(w) = I(m_t f)(w)\}) = 1$

■

Stochastic Integral

Proof of (1)

Since H_0^2 is dence in H^2, choose some functions $f_n \in H_0^2[0,T]$ s.t.

$$\| f - f_n \|_{L^2(dP\times dt)} \to 0$$

Define a new process $X_t^{(n)}$ by taking

$$X_t^{(n)}(w) := I(m_t f_n)(w)$$

By explicit formular, for some $k \in \mathbb{N}$ s.t. $t_k < t \le t_{k+1}$,

$$X_t^{(n)}(w) = a_k(w)\{B_t - B_{t_k}\} + \sum_{i=0}^{k-1} a_i(w)\{B_{t_{i+1}} - B_{t_i}\}$$

$$(\because\ m_t f_n(w,s) = \sum_{i=0}^{n-1} a_i(w) 1_{(t_i < t < t_{i+1})}(s) = a_k(w) 1_{(t_i < s < t)} + \sum_{i=0}^{k-1} a_i(w) 1_{(t_i < s < t_{i+1})})$$

Stochastic Integral

Since $X_t^{(n)}$ is a continuous F_t-adapted martingale, we can apply Doob's maximar ineqaulity tp the continuous submartingale $M_t := \left| X_t^{(n)} - X_t^{(m)} \right|$ for any $m \le n$ to find

$$P(\sup\left\{ \left| X_t^{(n)} - X_t^{(m)} \right| : 0 \le t \le T \right\} \ge \epsilon) \le \frac{1}{\epsilon^2} E\left(\left| X_t^{(n)} - X_t^{(m)} \right|^2 \right)$$

$$\le \frac{1}{\epsilon^2} \| f_n - f_m \|^2_{L^2(dP\times dt)} \quad (\because \text{ Ito's inequality})$$

Stochastic Integral

Because f_n converges to f in $L^2(dP\times dt)$, we can choose an increasing subsequence n_k s.t.

$$\max\left\{\| f_n - f_{n_k}\|^2_{L^2(dP\times dt)} : n_k \le n\right\} \le 2^{-3k}$$

Taking $\epsilon = 2^{-3k}$,

$$P\left(\sup\left\{\left| X_t^{(n_{k+1})} - X_t^{(n_k)}\right| : 0 \le t \le T\right\} \ge 2^{-k}\right) \le 2^{-k}$$

Stochastic Integral

We now use the next Lemma.

Lemma (Borel-Cantelli Lemma)

If $\{A_n\}$ is any sequence of events, then

$$\sum_{n=1}^{\infty} P(A_n) < \infty \text{ implies } P\left(\sum_{n=1}^{\infty} 1_{A_n} < \infty\right) = 1 \text{ i.e. } P\left(\sum_{n=1}^{\infty} 1_{A_n} = \infty\right) = 0$$

■

Taking $\Omega_0 = \bigcup_{n=1}^{\infty} A_n$ and $C(w) := \sum_{n=1}^{\infty} 1_{A_n}(w)$,

$$\sup\left\{\left|X_t^{(n_{k+1})} - X_t^{(n_k)}\right| : 0 \le t \le T\right\} \le 2^{-k} \text{ for all } k \ge C(w)$$

Therefore, for all $w \in \Omega_0$, $\left\{X_t^{(n_k)}(w)\right\}$ is a Cauchy sequence in the uniform norm on $C[0, T]$ and there is a continuous function $t \mapsto X_t(w)$ s.t.

$$X_t^{(n_k)}(w) \mapsto X_t(w) \text{ uniformly on } [0, T]$$

By construction, $\left\{X_t^{(n_k)}(w)\right\}$ is a F_t martingale, so the martingale identity for $\{X_t\}$ follows i.e. $\{X_t\}$ is a continuous martingale.

Stochastic Integral

Proof of (2)

By construction of f_{n_k},

$$m_t f_{n_k} \to m_t f \text{ in } L^2(dP \times dt)$$

By Ito's Isometry,

$$I(m_t f_{n_k}) \to I(m_t f) \text{ in } L^2(dP)$$

We already know that

$$X_t^{(n_k)}(w) \mapsto X_t(w) \text{ in } L^2(dP)$$

By the uniqueness of $L^2(dP)$ limits,

$$\| X_t - I(m_t f) \|_{L^2(dP)} = 0 \text{ for each } t \in [0, T]$$

This implies

$$P(\{w : X_t(w) = I(m_t f)(w)\}) = 1 \text{ for each } t \in [0, T]$$

■

Stochastic Integral

Notation (Ito integral sign)

If $f \in H^2[0, T]$ & $\{X_t\}_{t \in [0, T]}$ is a continuous-time martingale s.t.

$$P(\{w : X_t(w) = I(m_t f)(w)\}) = 1 \text{ for each } t \in [0, T],$$

then we write

$$\int_0^t f(w, s) dB_s := X_t(w)$$

for each $t \in [0, T]$.

■

Thanks !

Formula & Applications

Definition (Standard Process)

We say that a stochastic process $\{X_t\}_{0 \le t \le T}$ is a standard process provided that $\{X_t\}$ has the integral representation

$$\text{i.e. } X_t = X_0 + \int_0^t a(w,s)ds + \int_0^t b(w,s)dB_s \quad \text{for } 0 \le t \le T$$

where $a(\cdot,\cdot)$ & $b(\cdot,\cdot)$ are adapted, measurable processes satifying the conditions;

$$P\left(\int_0^T |a(w,s)|ds < \infty\right) = 1 \ \& \ P\left(\int_0^T |b(w,s)|^2 ds < \infty\right) = 1$$

■

Theorem (Ito's Formular for Standard Process)

Suppose

$$f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$$

$\{X_t\}$: a standard process having the integral representation

$$X_t = X_0 + \int_0^t a(w,s)ds + \int_0^t b(w,s)dB_s \text{ , for } 0 \le t \le T$$

Then we have

$$f(t,X_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)b^2(w,s)ds$$

In the language of the box caculus, for the process $Y_t = f(t,X_t)$, we have

$$dY_t = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} dX \bullet dX$$

■